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Convective cooling of a thin flat plate in laminar and turbulent flows

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Abstract—This paper deals with the analysis for the cooling of a flat plate in a convective flow, taking into account the longitudinal heat conduction through the plate. The energy balance equations reduce to a single one-parameter integro-differential equation, for the evolution of the temperature of the plate. An asymptotic analysis based on the multiple scale technique is used in order to obtain an analytical solution of the studied problem. Two time scales appear in the asymptotic limit of a very good conducting plate. At time $t = 0$, the plate at a temperature different from that of the fluid is placed parallel to the fluid stream. For large but finite plate thermal conductivity, a rapid transient generates and the temperature adjusts itself to a pseudo-equilibrium condition, needed for the slow further evolution of the plate temperature. Both laminar and turbulent boundary layer flows are considered and the three term asymptotic solution is compared with the numerical solution of the governing equation. A very good agreement is achieved even for values of the heat transfer parameter of the order of unity.

INTRODUCTION

THE STUDY of the coupled heat transfer processes between heat convection and heat conduction is very important because of the simultaneous effects in practical heat transfer processes. The effect of wall heat conduction on convective heat transfer, has been analysed in ref. [1]. Luikov [2] and Payvar [3] analysed the problem where the lower surface of a flat plate of finite thickness is maintained at a constant and uniform temperature. At the upper surface of the plate, heat is convected to a laminar boundary layer. Luikov [2] made two approximate solutions, one based on a differential analysis with low Prandtl number and the second based on an integral analysis with polynomial velocity and temperature profiles. He concluded that for Brun numbers larger than 0.1, the plate thermal resistance can be neglected. Payvar [3] used the Lighthill approximation [4] for large Prandtl numbers, to obtain an integral equation which has been solved numerically. He obtained asymptotic solutions for large and small Brun numbers. Axial conduction was not taken into account, in both these works. Heating (or cooling) of a flat plate in a convective flow was analysed by Sohal and Howell [5] and Karvinen [6]. In both of these works, the Lighthill approximation has been employed and an integro-differential equation for the plate temperature evolution has been derived. In ref. [5], a numerical technique has been used to solve this equation. Karvinen [6] used an iterative method to solve the integro-differential equation for both steady-state and transient cases. He obtained good agreement with experimental results. Perturbation techniques in order to solve analytically the integro-differential equation resulting from the externally heated flat plate in a convective flow were used in ref. [7]. This equation has only one parameter, α, defined as the ratio of the thermal resistance of the fluid to that of the plate. For large values of this parameter (very good conducting plate), a regular perturbation approach is applied using $1/\alpha$ as the small parameter of expansion. On the other hand, for very small values of this parameter, a singular perturbation technique has been employed (matched asymptotic expansion) in order to study the plate temperature evolution. The leading term of the expansion ($\alpha = 0$) has a self-similar solution and the integral equation has been numerically solved. For small values of α two boundary layers develop at both edges of the plate. These boundary layers, however, have only local effects and the leading order solution give accurate results for small values of α even for this singular problem. For $\alpha = 0$, the cooling process has been analysed in ref. [7]. However, for the case of large values of α the cooling process cannot be analysed using the same regular perturbation techniques used in that paper [7], because this series breaks down in the first-order terms.

The objective of the present work is to study this cooling (or heating) process, for both laminar and turbulent flow over a smooth surface, using multiple scale analysis due to the generation of rapid transients during the process.

FORMULATION

The physical model analysed is the following. A thin flat plate of length L, thickness $2h$ at initial temperature T_{p0} is placed parallel in a forced flow of an incompressible fluid with velocity U_∞ and temperature T_∞, at time $t = 0$. The thermal conductivity of the plate material enables heat conduction through the plate. Both edges of the plate are assumed to

NOMENCLATURE

c_g	specific heat of fluid
c_s	specific heat of the plate material
g	non-dimensional function introduced in equation (21)
h	half thickness of the plate
L	length of the plate
Nu	Nusselt number, $-q_c x/[(T_{p0}-T_\infty)\lambda_g]$
Pr	Prandtl number, $(\mu c/\lambda)_g$
q_c	heat flux transferred by convection
Re	Reynolds number of the flow, $U_\infty \rho_g L/\mu_g$
Re_c	critical Reynolds number
s	non-dimensional strained time defined after equation (14)
t	time
t_c	characteristic time defined in equation (10)
T_p	temperature of the plate
T_{p0}	initial temperature of the plate
T_∞	fluid temperature far from the plate
U_∞	fluid velocity far from the plate
x, y	Cartesian coordinates.

Greek symbols

α	non-dimensional parameter defined in equation (9)
α_L	value of α for laminar flow
α_T	value of α for turbulent flow
ε	small parameter of expansion defined in equation (12)
η	non-dimensional transversal coordinate defined in equation (2)
θ_p	non-dimensional plate temperature defined in equation (2)
λ_g	thermal conductivity of the fluid
λ_s	thermal conductivity of the plate material
μ_g	viscosity coefficient of the fluid
ρ_s	density of the plate material
σ	non-dimensional fast time variable defined after equation (14)
τ	non-dimensional time defined in equation (2)
τ_L	non-dimensional time for laminar flow
τ_T	non-dimensional time for turbulent flow
χ	non-dimensional longitudinal coordinate defined in equation (2)
χ_c	position of the transition region
ω_i	constants introduced after equation (14).

be adiabatic, for simplicity. For small times, heat is transferred to the fluid in a uniform way. As the thermal boundary layer develops, a temperature gradient is produced in the plate, thus conducting heat upstream. Finally, this heat is transferred to the fluid in regions close to the leading edge. The energy balance equation in the plate is

$$\partial^2 T_p/\partial x^2 + \partial^2 T_p/\partial y^2 = \rho_s c_s/\lambda_s \, \partial T_p/\partial t \quad (1)$$

where T_p corresponds to the plate temperature; x and y correspond to the Cartesian coordinates longitudinal and transversal, respectively. The origin is located in the upper surface of the plate at the leading edge. The time is designed by t. ρ_s, c_s and λ_s correspond to the density, specific heat and the thermal conductivity, respectively. The boundary and initial conditions are given by

$$T_p = T_{p0} \text{ at } t = 0; \quad \partial T_p/\partial x = 0 \text{ at } x = 0 \text{ and } L;$$
$$\lambda_s \partial T_p/\partial y = -q_c \text{ at } y = 0; \quad \partial T_p/\partial y = 0 \text{ at } y = -h$$

where q_c corresponds to the heat convected to the fluid in the upper surface. Introducing the following non-dimensional variables:

$$\theta_p = (T_p - T_\infty)/(T_{p0} - T_\infty); \quad \chi = x/L, \quad \eta = y/h; \quad \tau = t/t_c \quad (2)$$

equation (1) takes the form

$$(h/L)^2 \partial^2\theta_p/\partial\chi^2 + \partial^2\theta_p/\partial\eta^2 = [\rho_s c_s h^2/(\lambda_s t_c)]\, \partial\theta_p/\partial\tau. \quad (3)$$

The characteristic time t_c is defined later. Due to the fact that the non-dimensional temperature of the plate is a function of $\theta_p = \theta_p(\chi, \eta, \tau)$, it is convenient to introduce a simplification that makes possible an analytical solution of the studied physical problem. We suppose that the temperature of the plate is a function of the longitudinal coordinate χ and time τ, in a first approximation. The restrictions associated with this approximation are pointed out later. In this case, we assume that θ_p is given by

$$\theta_p(\chi, \eta, \tau) = \theta_0(\chi, \tau) + \varepsilon\theta_1(\chi, \eta, \tau) + O(\varepsilon^2) \quad (4)$$

where ε is a small number compared with unity, to be defined later. Introducing relation (4) in equation (3), this equation transforms to

$$(h/L)^2[\partial^2\theta_0/\partial\chi^2 + \varepsilon\partial^2\theta_1/\partial\chi^2 + \cdots] + \varepsilon\partial^2\theta_1/\partial\eta^2 + \varepsilon^2\partial^2\theta_2/\partial\eta^2 + \cdots = [\rho_s c_s h^2/(\lambda_s t_c)]\{\partial\theta_0/\partial\tau + \varepsilon\partial\theta_1/\partial\tau + \cdots\}. \quad (5)$$

Integrating equation (5) in the form

$$\int_0^1 [\;]\, d\eta$$

and neglecting terms of higher order, we obtain in a first approximation

$$(h/L)^2\partial^2\theta_0/\partial\chi^2 - (Nu/\chi)(\lambda_g/\lambda_s)(h/L) = \rho_s c_s h^2/(\lambda_s t_c)\, \partial\theta_0/\partial\tau. \quad (6)$$

In this equation, both boundary conditions at $y = 0$

and $-h$ have been applied. The Nusselt number Nu is given by

$$Nu = -q_c x/[(T_{p0}-T_\infty)\lambda_g]$$

where λ_g represents the fluid thermal conductivity. Due to the fact that the characteristic times in the fluid are in general very small compared with the characteristic times in the solid, we can assume the quasi-steady approximation in the fluid phase. The solution of the energy equation in the fluid can be obtained using the asymptotic Lighthill approximation, derived for the case of small thermal boundary layer thickness compared with the momentum boundary layer thickness. The Nusselt number can be given as

$$Nu = a\, Pr^m\, Re^n\, \chi^n \left[\theta_1 + \int_{\theta_1}^{\theta} K(\chi,\chi')\,\mathrm{d}\theta'\right] \quad (7)$$

where Pr is the Prandtl number, $Pr = \mu_g c_g/\lambda_g$, and Re the Reynolds number based on the length of the plate, $Re = U_\infty \rho_g L/\mu_g$. The subscript 1 represents the conditions at the leading edge of the plate. The kernel of the integral in equation (7) is given by

$$K(\chi,\chi') = \{1-(\chi'/\chi)^b\}^{-c}.$$

The constants a, b, c, n and m depend on the flow characteristics. Table 1 gives the values for the laminar and turbulent boundary layer flows over a smooth plate.

Introducing equation (7) in equation (6) gives

$$\alpha\,\partial^2\theta/\partial\chi^2 = \partial\theta/\partial\tau + \left[\theta_1 + \int_{\theta_1}^{\theta} K(\chi,\chi')\,\mathrm{d}\theta'\right]\Big/(\chi)^{1-n}. \quad (8)$$

Here, the subscript 0 has been taken out for simplicity. The integro-differential equation (8) gives the evolution of the temperature of the plate and contains only one parameter α defined as

$$\alpha = (h/L)(\lambda_s/\lambda_g)(a\, Pr^m\, Re^n)^{-1}. \quad (9)$$

This parameter represents the relation between the heat conducted through the plate to that convected to the fluid. For $\alpha \gg 1$ the heat conducted through the plate is very large, thus not admitting large temperature longitudinal gradients. On the other side, for $\alpha \ll 1$, the heat convection to the fluid is the most important. The characteristic time t_c then is given by

$$t_c = hL\rho_s c_s[a\lambda_g\, Re^n\, Pr^m]^{-1}. \quad (10)$$

Table 1

	Laminar flow	Turbulent flow
a	0.332	0.0287
b	3/4	9/10
c	1/3	1.9
m	1/3	3/5
n	1/2	8/10

Integrating equation (5) now in the form

$$\int_0^1 [\;]\,\mathrm{d}\chi$$

this equation reduces in a first approximation

$$\varepsilon\partial^2/\partial\eta^2\left\{\int_0^1 \theta_1\,\mathrm{d}\chi\right\} = \rho_s c_s h^2/\lambda_s\,\mathrm{d}\tau/\mathrm{d}t\,\partial/\partial\tau\left\{\int_0^1 \theta_0\,\mathrm{d}\chi\right\}. \quad (11)$$

From this relation we can obtain the definition for ε as

$$\varepsilon = a(h/L)(\lambda_g/\lambda_s)\, Re^n\, Pr^m \ll 1. \quad (12)$$

Equation (12) gives the restriction in using the approximation introduced previously. From equations (9) and (12), the value of α has to be much greater than $(h/L)^2$ which gives us a wide range in the validity of the approximation introduced. The initial and boundary conditions are given by

$$\theta(\chi,0) = 1 \quad \text{and} \quad \partial\theta/\partial\chi = 0 \quad \text{at} \quad \chi = 0, 1. \quad (13)$$

The temperature of the plate is then $\theta = F(\chi,\tau,\alpha)$. The solution of the integro-differential equation can be obtained in the asymptotic limits, $\alpha \gg 1$ and $\alpha \ll 1$. In the following section, the asymptotic limit $\alpha \gg 1$ is obtained

ASYMPTOTIC LIMIT $\alpha \gg 1$

For large values of α the plate temperature varies little in the streamwise direction. A regular expansion of the form

$$\theta = \theta_0(\tau) + \sum_{j=1}^{\infty} 1/\alpha^j \theta_j(\chi,\tau)$$

breaks down in terms of the order of $1/\alpha$, because the appearance of transients of the order of $1/\alpha$ in the non-dimensional time. The solution can be obtained using the multiple scale analysis, assuming the following expansion:

$$\theta = \theta_0(s) + \sum_{j=1}^{\infty} 1/\alpha^j \theta_j(\chi,s,\sigma) \quad (14)$$

where

$$s = \tau(1+\omega_1/\alpha+\omega_2/\alpha^2\cdots); \quad \sigma = \alpha\tau$$

represent the two time scales presented in the problem. The constants ω_i appear in order to cancel the secular terms arising in the solution. Introducing equation (14) into equation (8) we obtain the following set of equations:

$$\partial^2\theta_0/\partial\chi^2 - \partial\theta_0/\partial\sigma = 0 \quad (15)$$

$$\partial^2\theta_{j+1}/\partial\chi^2 - \partial\theta_{j+1}/\partial\sigma = \partial\theta_j/\partial s + \omega_1\partial\theta_{j-1}/\partial s + \cdots + \left[\theta_{j1} + \int_{\theta_{j1}}^{\theta_j} K(\chi,\chi')\,\mathrm{d}\theta_j'\right]\Big/(\chi)^{1-n} \text{ for all } j. \quad (16)$$

The boundary conditions are given by

$$\theta_0(\chi,0,0) = 1;\quad \theta_j(\chi,0,0) = 0 \quad \text{for} \quad j>0$$

$$\partial\theta_j/\partial\chi = 0 \quad \text{at} \quad \chi = 0,1 \quad \text{for all} \quad j.$$

The solution of equation (15) together with the boundary conditions gives that $\theta_0 = \theta_0(s)$. The first equation in equation (16) is given by

$$\partial^2\theta_1/\partial\chi^2 - \partial\theta_1/\partial\sigma = \mathrm{d}\theta_0/\mathrm{d}s + \theta_0/(\chi)^{1-n}. \quad (17)$$

Integrating equation (17) in the form

$$\int_0^1 [\;]\,\mathrm{d}\chi$$

and applying the adiabatic boundary conditions at both edges of the plate, we obtain

$$-\partial/\partial\sigma \int_0^1 \theta_1\,\mathrm{d}\chi = \mathrm{d}\theta_0/\mathrm{d}\tau + (1/n)\theta_0. \quad (18)$$

The equality has to be zero in order not to allow the appearance of secular terms. Applying the initial condition for θ_0, the solution of equation (18) is given by

$$\theta_0 = \exp(-\tau/n). \quad (19)$$

Below we will concentrate on the laminar case, giving only the final results for the turbulent case. For the laminar boundary layer, equation (17) takes the form

$$\partial^2\theta_1/\partial\chi^2 - \partial\theta_1/\partial\sigma = \theta_0[1/(\chi)^{1/2} - 2]. \quad (20)$$

The initial and boundary conditions for θ_1 are given by

$$\theta_1(\chi,0,0) = 0;\quad \partial\theta_1/\partial\chi = 0 \quad \text{at} \quad \chi = 0,1.$$

In order to reduce equation (18) to a homogeneous differential equation it is convenient to introduce the function $g(\chi,\sigma)$ in the following way:

$$\theta_1 = \theta_0[4/3\chi^{3/2} - \chi^2 + g(\chi,\sigma)]. \quad (21)$$

The equation for g is then given by

$$\partial^2 g/\partial\chi^2 - \partial g/\partial\sigma = 0 \quad (22)$$

with the initial and boundary conditions

$$g(\chi,\sigma=0) = \chi^2 - 4/3\chi^{3/2};$$

$$\partial g/\partial\chi = 0 \quad \text{for} \quad \chi = 0,1. \quad (23)$$

The solution of equations (22) and (23) can be obtained with the help of the Fourier transform method, giving

$$g(\chi,\sigma) = \sum_{j=0}^{\infty} a_j \cos(\pi j\chi)\exp(-j^2\pi^2\sigma) \quad (24)$$

with the Fourier coefficients, a_j given by

$$a_j = b_j \int_0^1 [\chi^2 - 4/3\chi^{3/2}]\cos(j\pi\chi)\,\mathrm{d}\chi$$

where $b_0 = 1$ and $b_j = 2$ for $j>0$. Evaluating these coefficients, the solution for θ_1 is therefore

$$\theta_1(\chi,s,\sigma) = \left[-1/5 + 4/3\chi^{3/2} - \chi^2 + (2\pi)^{1/2} \times \sum_{j=1}^{\infty} \cos(\pi j\chi)\exp(-j^2\pi^2\sigma)/(j\pi)^{5/2}\right]\exp(-2s). \quad (25)$$

The equation for θ_2 is

$$\begin{aligned}
&\partial^2\theta_2/\partial\chi^2 - \partial\theta_2/\partial\sigma \\
&= -2\theta_0\Big\{\omega_1 - 1/5 + 4/3\chi^{3/2} - \chi^2 + \sqrt{(2\pi)} \\
&\quad \times \sum_{j=1}^{\infty} \cos(\pi j\chi)\exp(-j^2\pi^2\sigma)/(j\pi)^{5/2} \\
&\quad - 1/\sqrt{(4\chi)}\Big[-1/5 + \sqrt{(2\pi)} \\
&\quad \times \sum_{j=1}^{\infty} \exp(-j^2\pi^2\sigma)/(j\pi)^{5/2}\Big]\Big\} + \theta_0/\sqrt{\chi}\int_0^{\chi} K(\chi,\chi') \\
&\quad \times \Big[-2\chi' + 2\sqrt{\chi'} - \sqrt{(2\pi)}\sum_{j=1}^{\infty}\sin(\pi j\chi) \\
&\quad \times \exp(-j^2\pi^2\sigma)/(j\pi)^{3/2}\Big]\mathrm{d}\chi'.
\end{aligned} \quad (26)$$

Integrating equation (26) in the streamwise direction from the leading edge to the trailing edge and applying the adiabatic boundary conditions at both edges, we obtain

$$\begin{aligned}
&-\partial/\partial\sigma \int_0^1 \theta_2\,\mathrm{d}\chi = \\
&-2\theta_0\{\omega_1 + 1/5 + 8/15\beta(8/3,2/3) - 2/3\beta(2,2/3)\} \\
&-\sqrt{(2\pi)}\theta_0 \sum_{j=1}^{\infty}\exp(-j^2\pi^2\sigma)/(j\pi)^{3/2} \\
&\times\Big\{-4/(3j\pi)\int_0^1 \sqrt{\chi'}\cos(j\pi\chi')\,\mathrm{d}\chi' + 1/(j\pi) \\
&+ 1/(4j\pi)\int_0^1 \chi^{-5/4}\Big[\int_0^1 \chi'^{-1/4}\cos(j\pi\chi')\,\mathrm{d}\chi'\Big]\mathrm{d}\chi\Big\}.
\end{aligned} \quad (27)$$

Secular terms arise for the non-vanishing value between brackets in the first term on the right-hand side of equation (27). Therefore, ω_1 is

$$\omega_1 = -1/5 - 8/15\beta(8/3,2/3) + 2/3\beta(2,2/3) \approx +0.009893.$$

Figure 1 shows $\theta_2 e^{2s}$ as obtained from the numerical solution of equation (26), for different values of the non-dimensional time σ. In the case of turbulent flow, it can be shown, following the same procedure, that the non-dimensional temperature of the plate up to

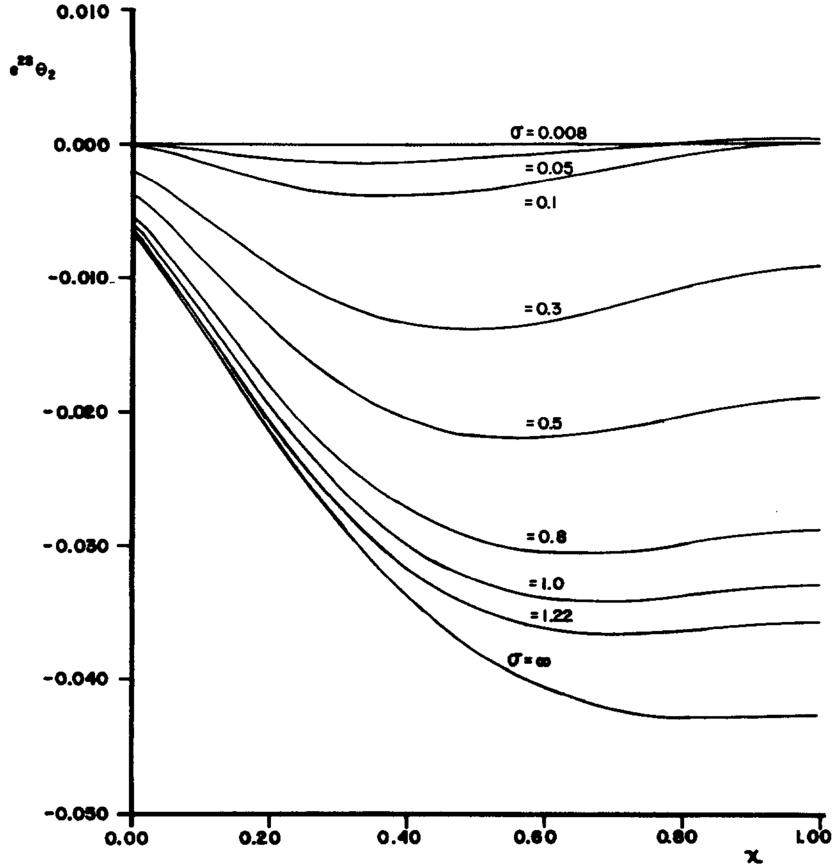

FIG. 1. Evolution of the second-order term, θ_2/θ_0, as a function of χ for different values of the non-dimensional fast time σ.

first order, can be written as

$$\theta(\chi, s, \sigma) = \exp(-5s/4)\left\{1 + 1/\alpha\left[-5/8\chi^2 + 25/36\chi^{9/5} - 5/126 + 0.71953\sum_{j=1}^{\chi}\cos(\pi j \chi) \times \exp(-j^2\pi^2\sigma)/(j\pi)^{14/5}\right]\right\} + O(1/\alpha^2) \quad (28)$$

and $\omega_1 \approx 0.00072672$. It is to be noted that α, s and σ are defined differently in both cases, laminar and turbulent boundary layer. In the next section we will present the numerical calculations obtained by solving the integro-differential equation and compare it with the asymptotic solution deduced in this section.

RESULTS AND DISCUSSION

For large values of α, that is $\alpha \to \infty$, equation (19) gives the non-dimensional temperature of the plate as a function of the non-dimensional time. Due to the fact that α and the characteristic time are not defined in the same way, for laminar and turbulent boundary layers, it is convenient to relate one to each other as

$$\alpha_T = [11.56794 Pr^{-4/15} Re^{-3/10}]\alpha_L$$

where α_T and α_L, correspond to the values of α for turbulent and laminar flow, respectively. In the same form, we can relate the non-dimensional times τ used for both laminar and turbulent flows as

$$\tau_T = [0.086445 Pr^{4/15} Re^{3/10}]\tau_L.$$

In this asymptotic limit, $\alpha \to \infty$, the ratio of the temperatures of the plate cooled by turbulent and laminar flows is given from equation (19) as

$$\theta_T/\theta_L = \exp\{2 - 0.1080562 Pr^{4/15} Re^{3/10}\}\tau_L. \quad (29)$$

Figure 2 shows this ratio as a function of the non-dimensional time τ defined for laminar flow (τ_L), for different Reynolds numbers and Prandtl number of unity. Of course this is valid if the whole boundary layer is either laminar or turbulent. As the Reynolds number increases, the plate temperature decreases faster in comparison with the laminar case. If the Reynolds number is only slightly larger than the critical Reynolds number of the flow, both types of flow structures coexist—laminar flow upstream of the transition zone and turbulent flow downstream. Assuming this transition zone to be very small compared with the length of the plate, in the limit $\alpha \to \infty$ we have that the governing equation is given by

$$\alpha_L \partial^2\theta/\partial\chi^2 = \partial\theta/\partial\tau_L + \theta/\sqrt{x}, \text{ for } 0 < \chi < \chi_c \quad (30)$$

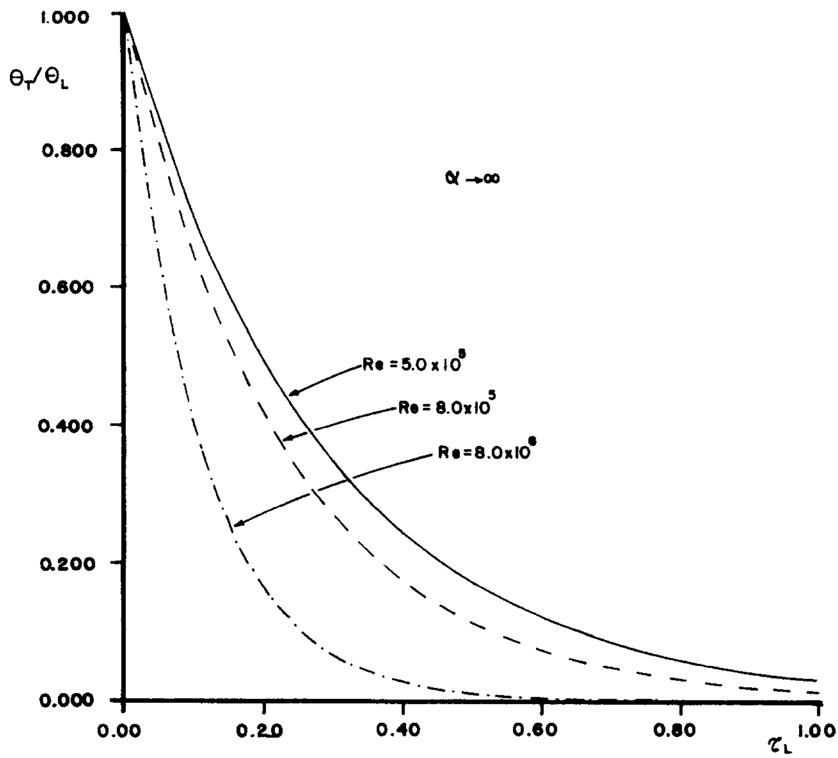

FIG. 2. Non-dimensional temperature ratio (turbulent/laminar) for different Reynolds numbers and $Pr = 1$, in the limit $\alpha \to \infty$.

$$\alpha_T \partial^2\theta/\partial\chi^2 = \partial\theta/\partial\tau_T + \theta/\chi^{2/10}, \text{ for } \chi_c < \chi < 1 \quad (31)$$

where χ_c corresponds to the position of the transition region ($Re = Re_c$). Integrating equation (30) in the form

$$\int_0^{\chi_c} [\;] d\chi$$

and equation (31) in the form

$$\int_{\chi_c}^{1} [\;] d\chi$$

and applying the adiabatic boundary conditions at both edges, we obtain

$$\alpha_L \partial\theta/\partial\chi|_{\chi_c} = \partial\theta/\partial\tau_L \chi_c + 2\theta\sqrt{\chi_c} \quad (32)$$

$$-\alpha_T \partial\theta/\partial\chi|_{\chi_c} = \partial\theta/\partial\tau_T(1-\chi_c) + 10/8\theta(1-\chi_c^{8/10}). \quad (33)$$

Because of the continuity of the plate temperature and the heat flux at the position of the transition zone, we obtain from equations (32) and (33), the evolution equation for the plate temperature as

$$d\theta/d\tau_L[\chi_c + (\alpha_L/\alpha_T)(\tau_L/\tau_T)(1-\chi_c)] = -\theta[2\sqrt{\chi_c} + 10/8(\alpha_L/\alpha_T)(1-\chi_c^{8/10})]$$

the solution to which is given finally by

$$\theta = \exp\{-[2\sqrt{\chi_c} + 0.1080572 Pr^{4/5} \times Re^{3/10}(1-\chi_c^{8/10})]\tau_L\}. \quad (34)$$

For $\chi_c = 0$ we recover the pure turbulent flow and for $\chi_c = 1$ the laminar flow solution is reproduced.

Figure 3 shows the non-dimensional temperature profiles as a function of χ, for three different values of α, for $\tau = 0.05$, for the laminar flow. For $\alpha = 100$, the temperature is uniform and the leading term in the asymptotic expansion gives the temperature evolution. However, for values of $\alpha = 1$, there is an important temperature difference between the leading and trailing edges of about 12%, due to the finite thermal conductivity of the plate material. For $\alpha = 5$, a temperature difference, between both edges, of about 5% is found from the two-term asymptotic solution given by equations (19) and (25). The corresponding solution to turbulent flow is given in Fig. 4. Though the trends are found to be qualitatively the same for both laminar and turbulent flows, the crossing point of the non-dimensional temperature profiles with that given for large α (or α_T), is shifted to the leading edge in the case of turbulent flow, thus indicating that the effect of the thermal conductivity of the plate is larger for turbulent flow. For larger values of α_T, the cooling effect is more effective for turbulent flow.

Figure 5 shows the comparison of the two-term asymptotic expansion with the numerical calculation of the integro-differential equation, for $\alpha = 5$ and different non-dimensional times, for the case of a laminar flow. The numerical technique used for solving the integro-differential equation (8), is described elsewhere [8]. The asymptotic analysis gives very good results for values of $\alpha = 5$. The error introduced in

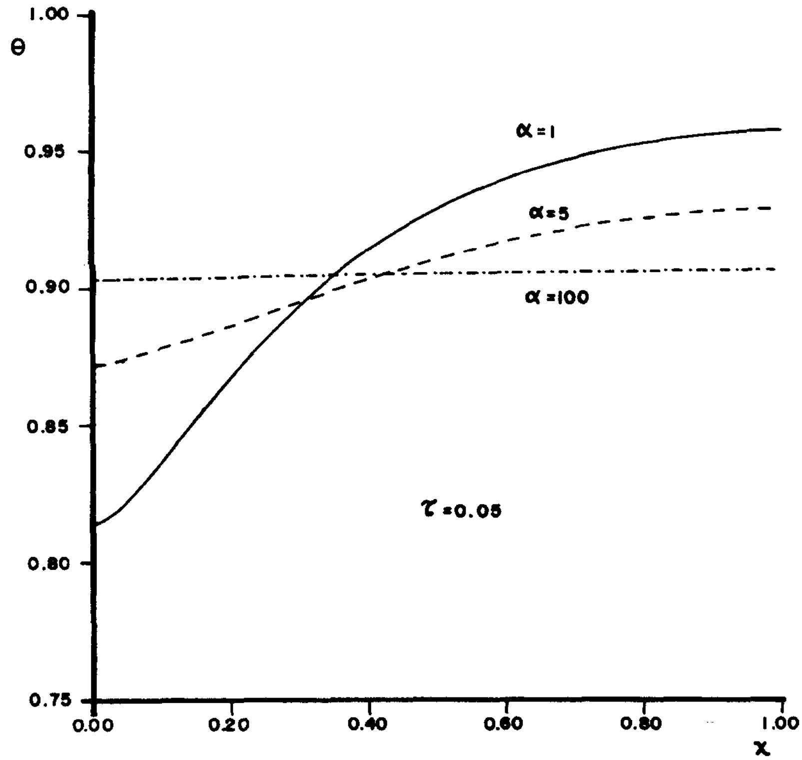

FIG. 3. Non-dimensional temperature profiles for different values of α at $\tau = 0.05$ (laminar flow), as obtained with the two-term asymptotic expansion.

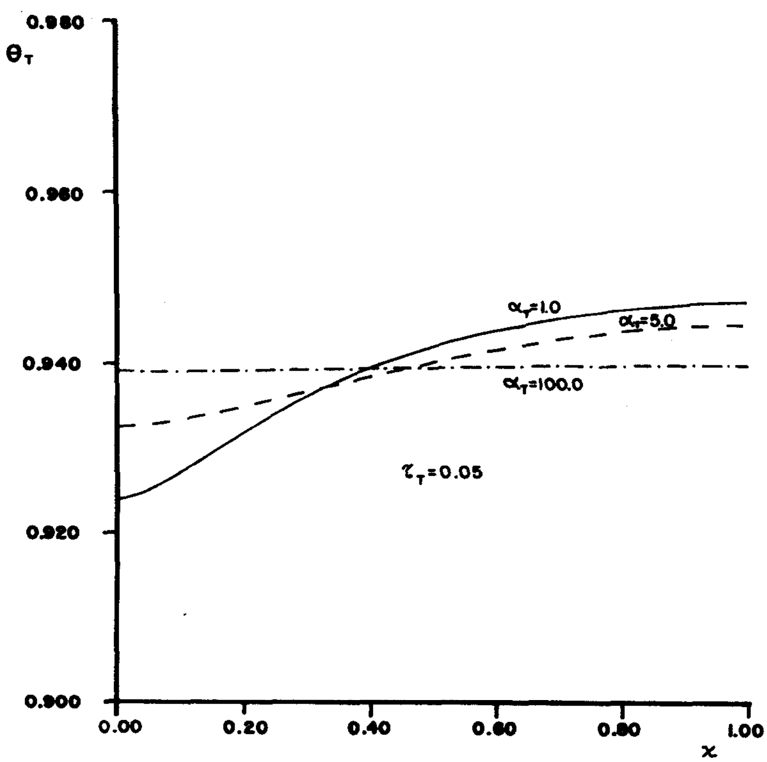

FIG. 4. Non-dimensional temperature profiles for different values of α_T at $\tau_T = 0.05$ (turbulent flow).

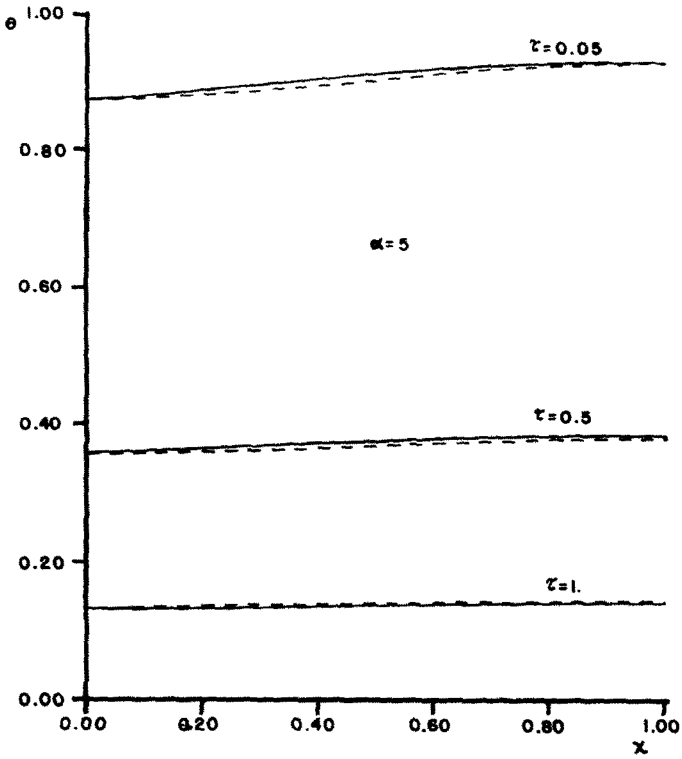

FIG. 5. Non-dimensional temperature profiles at different times, for $\alpha = 5$ (laminar flow): ———, asymptotic solution; - - - - - -, numerical solution.

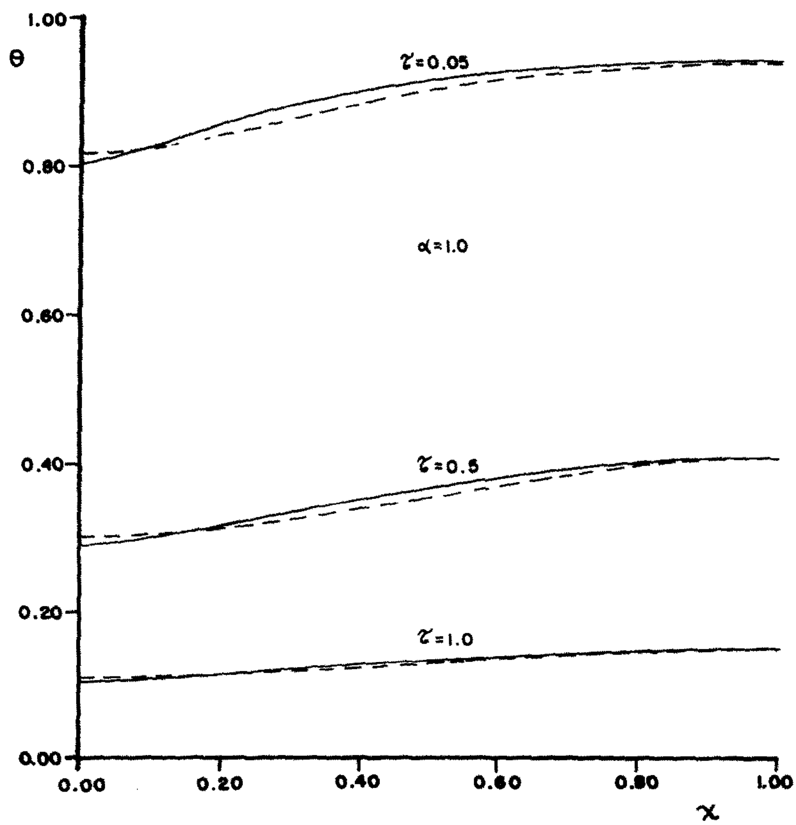

FIG. 6. Non-dimensional temperature profiles at different times, for $\alpha = 1$ (laminar flow): ———, asymptotic solution; - - - - - -, numerical solution.

this case is not larger than 3%. The case of $\alpha = 1$ is shown in Fig. 6. In this case, the errors introduced are much larger than in the previous case, giving errors of the order of 6%. A new term in the asymptotic expansion is needed for values of α close to unity. Figures 7 and 8 show the influence of the third term in the asymptotic expansion on the temperature evolution for $\alpha = 1$ and two different times. It is shown here that the three-term asymptotic expansion reproduces the numerical solution in portions of the middle of the plate. Still a difference is noted at both edges of the plate. Figure 9 shows the non-dimensional temperatures at both edges as a function of the non-dimensional time, for $\alpha = 5$ and laminar flow. This temperature difference between both edges, remains of the same order as τ increases. For values of $\tau > 3$, practically the temperature of the plate reaches the stream temperature. Finally, Figs. 10 and 11 show the non-dimensional temperature evolution in the trailing edge of the plate for two different values of α_L (5,100), exposed to a turbulent boundary layer flow, for different Reynolds numbers and a Prandtl number of unity.

CONCLUSIONS

In this paper, the cooling process of a flat plate in a convective flow, has been analysed using asymptotic techniques. The finite thermal conductivity of the plate material allows heat upstream to be conducted through the plate, thus changing the mathematical character of the problem, to an elliptic one. Assuming the plate to have adiabatic leading and trailing edges, the overall heat convection through the upper surface of the plate governs the evolution of the plate temperature with time and space. The governing equations reduce to a single-parameter integro-differential equation. The asymptotic limit of large values for this parameter has been explored here. At time equal to zero, a heated plate is placed parallel to a fluid flow, thus generating a large Reynolds number flow. If the initial temperature of the plate is different from that of the fluid, heat is transferred via convection from the plate to the fluid at an almost uniform rate through the plate, at the beginning. Once the thermal boundary layer develops, temperature gradients appear in the plate, thus making possible the heat conduction through the plate. For non-dimensional times much smaller than unity, a rapid transient process represented by the transition from a uniform temperature profile to a pseudo-equilibrium state, takes place. Once this pseudo-equilibrium temperature distribution is established, the whole temperature of the plate decreases at a slow rate, reaching asymptotically the uniform temperature distribution equal to the fluid temperature. For large values of α, these transient processes at the beginning take place in a short time scale. A three-term asymptotic expansion for

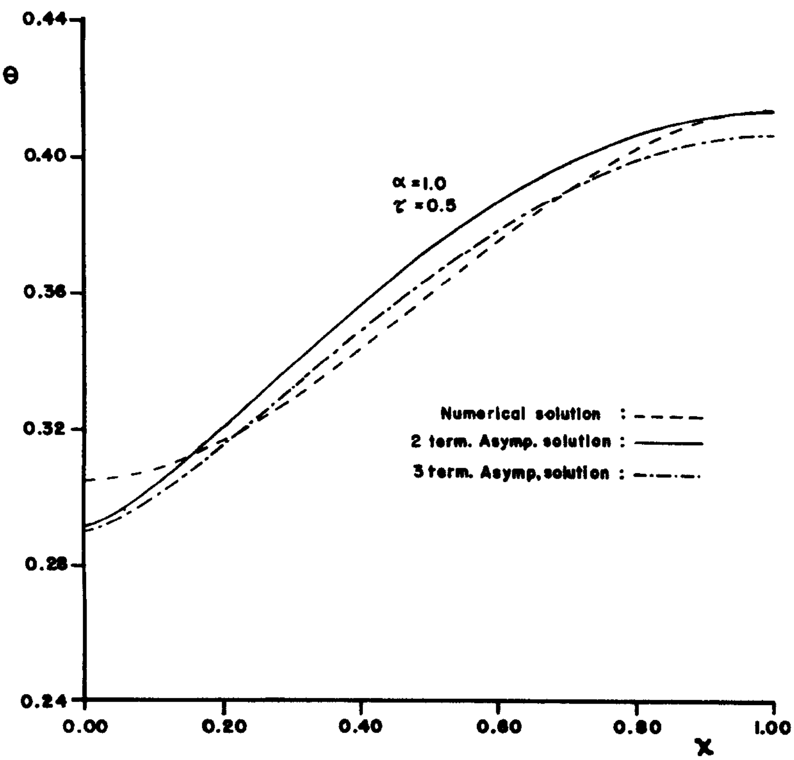

FIG. 7. Non-dimensional temperature profiles for $\alpha = 1$ and $\tau = 0.5$ as obtained by: ———, two-term asymptotic expansion; – – – –, three-term asymptotic expansion; - - - - - - -, numerical solution.

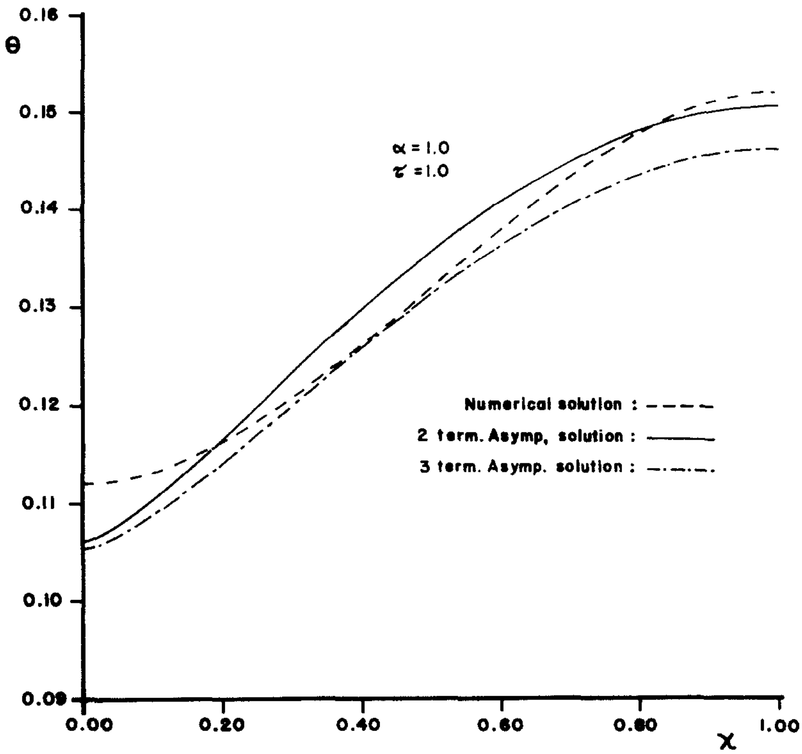

FIG. 8. Non-dimensional temperature profiles for $\alpha = 1$ and $\tau = 1.0$ as obtained by: ———, two-term asymptotic expansion; — — — —, three-term asymptotic expansion; - - - - - - - -, numerical solution.

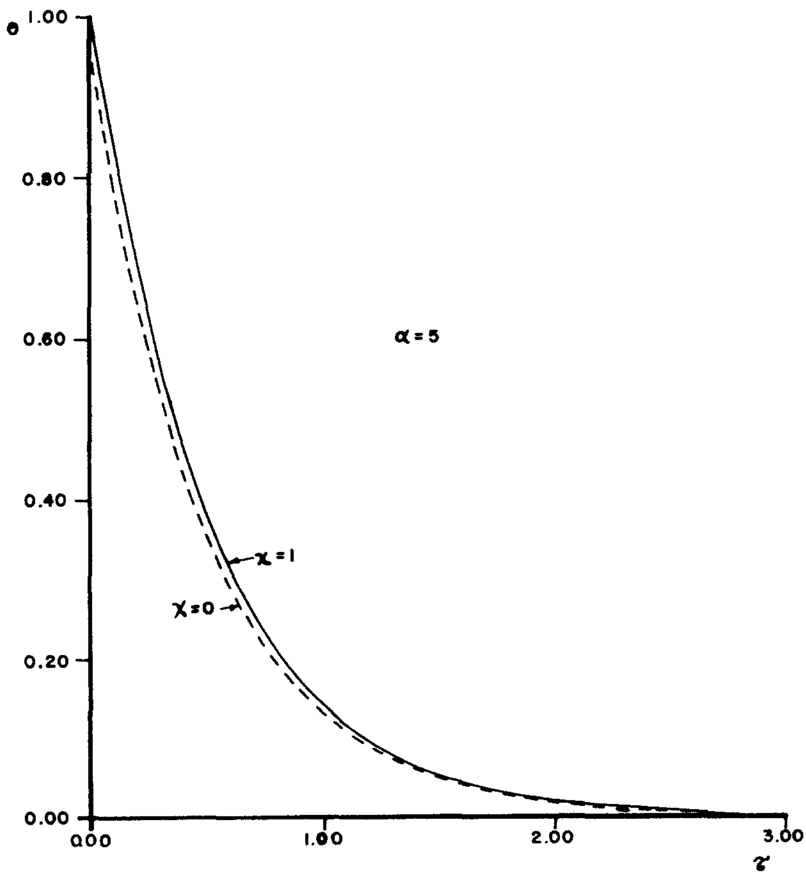

FIG. 9. Non-dimensional temperature evolution at both edges for $\alpha = 5$ (laminar flow).

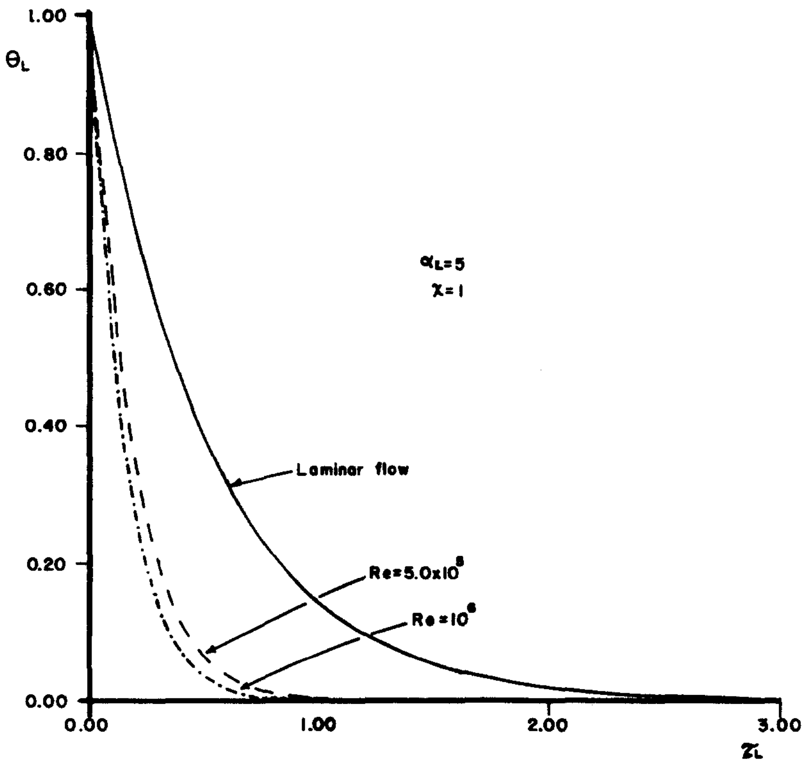

FIG. 10. Non-dimensional temperature at the trailing edge of the plate for $\alpha_L = 5$, for different Reynolds numbers and $Pr = 1$ (turbulent flow).

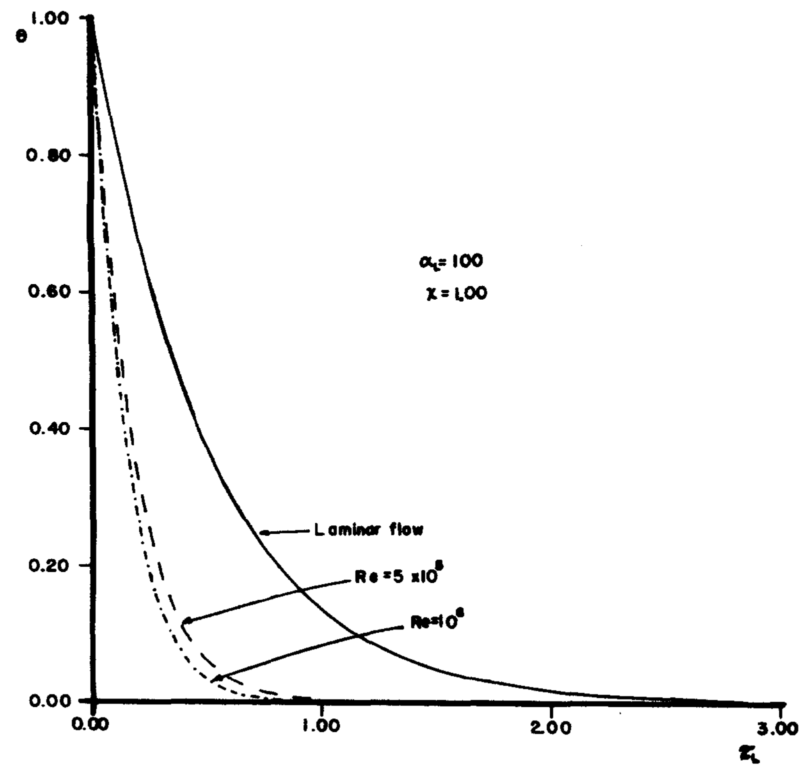

FIG. 11. Non-dimensional temperature at the trailing edge of the plate for $\alpha_L = 100$, for different Reynolds numbers and $Pr = 1$ (turbulent flow).

the temperature of the plate is deduced, using the multiple-scale technique. This analytical solution for the temperature evolution is compared with the numerical solution of the integro-differential equation, giving very good agreement for values of α close to 5 or larger. For values of α close to unity, 6% error is found. The analysis has been carried out for both laminar and turbulent (on smooth plate surface) flows. The solution shows that the parameter α has a bigger influence in turbulent than in laminar flows.

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REFROIDISSEMENT CONVECTIF D'UNE PLAQUE PLANE ET MINCE DANS DES ECOULEMENTS LAMINAIRES OU TURBULENTS

Résumé—On analyse le refroidissement d'une plaque plane dans un écoulement en prenant en compte la conduction longitudinale à travers la plaque. Les équations de bilan d'énergie se réduisent à une équation intégro-différentielle à un seul paramètre pour l'évolution de la température de la plaque. Une analyse asymptotique basée sur la technique d'échelle multiple est utilisée pour obtenir une solution analytique du problème étudié. Deux échelles de temps apparaissent dans la limite asymptotique d'une plaque parfaitement conductrice. Au temps $t = 0$, la plaque à une température différente de celle du fluide est placée parallèlement à l'écoulement. Pour une conductivité grande mais finie de la plaque, il se produit un transitoire rapide et la température s'ajuste à une condition de pseudo-équilibre suivie d'une évolution lente de la température de la plaque. On considère les écoulements de couche limite laminaires et turbulents et la solution asymptotique à trois termes est comparée à la solution numérique. Un très bon accord est constaté, même pour des valeurs du paramètre de transfert thermique de l'ordre de l'unité.

KONVEKTIVE KÜHLUNG EINER DÜNNEN EBENEN PLATTE IN LAMINARER UND TURBULENTER STRÖMUNG

Zusammenfassung—Die Abkühlung einer Platte in konvektiver Strömung wird untersucht, wobei die Längswärmeleitung in der Platte berücksichtigt wird. Für die Berechnung des Temperaturverlaufs in der Platte werden die Energiebilanzgleichungen auf eine einzige einparametrige Integral-Differentialgleichung reduziert. Um eine analytische Lösung des Problems zu erhalten, verwendet man eine asymptotische Näherung für unterschiedliche Zeiten. Zwei Zeitbereiche beschreiben das asymptotische Verhalten einer sehr gut wärmeleitenden Platte. Bei $t = 0$ wird parallel zur Strömung eine Platte eingebracht, deren Temperatur sich von derjenigen der Flüssigkeit unterscheidet. Bei großer, aber endlicher Wärmeleitfähigkeit der Platte ergibt sich eine schnelle Temperaturänderung und anschließend ein Pseudo-Gleichgewichts-Zustand, welcher der weiteren langsamen Temperaturentwicklung zugrundegelegt wird. Sowohl laminare als auch turbulente Grenzschichtströmung wird berücksichtigt, und die Lösung dritter Ordnung wird mit der numerischen Lösung des problems verglichen. Eine sehr gute Übereinstimmung wird sogar dann erreicht, wenn der Wärmeübergangsfaktor in der Größenordnung von eins liegt.

КОНВЕКТИВНОЕ ОХЛАЖДЕНИЕ ТОНКОЙ ПЛОСКОЙ ПЛАСТИНЫ В ЛАМИНАРНЫХ И ТРБУЛЕНТНЫХ ПОТОКАХ

Аннотация—Анализируется процесс охлаждения плоской пластины в конвективном потоке с учетом продольной теплопроводности в ней. Для описания изменений температуры пластины уравнения сохранения энергии приводятся к однопараметрическому интегро-дифференциальному уравнению. Для получения аналитического решения задачи используется асимптотический анализ на основе метода множественных масштабов. В асимптотическом пределе пластины с хорошей теплопроводностью появляются два временных масштаба. При значении времени $t = 0$ пластина с температурой, отличной от температуры жидкости, устанавливается параллельно потоку жидкости. В случае больших, но конечных значений коэффициента теплопроводности пластины возникает быстро протекающая неустойчивость, и изменение температуры приходит к квазиравновесному режиму с характерным медленным ее изменением. Рассматриваются как ламинарное, так и турбулентное течения в пограничном слое, и проводится сравнение трехчленного асимптотического решения с численным решением определяющего уравнения. Получено очень хорошее соответствие даже для значений параметра теплопереноса порядка единицы.